1. You draw 4 balls without replacement from an urn with 3 blue balls and 3 red balls. Let $X$ be the number of blue balls in the draw.
(a) Find the PMF and expected value of $X$.

Solution: We represent the drawn balls by the first four in a random arrangement of 3 blue and 3 red balls. There are $\binom{6}{3}=20$ arrangements, out of which 4 contain exactly 1 blue ball in the first 4 positions, so by the equally likely outcomes formula, $\mathrm{P}(X=1)=4 / 20=1 / 5$. By symmetry of the red and blue balls $\mathrm{P}(X=3)=1 / 5$ and by the axioms of probability $\mathrm{P}(X=2)=1-2 \cdot 1 / 5=3 / 5$. The expected value is

$$
\mathrm{E}[X]=\mathrm{P}(X=1)+2 \mathrm{P}(X=2)+3 \mathrm{P}(X=3)=\frac{1}{5}+2 \cdot \frac{3}{5}+3 \cdot \frac{1}{5}=2
$$

(b) Find the variance of $X$.

Solution: $\operatorname{Var}(X)=\mathrm{E}\left[(X-\mathrm{E}[X])^{2}\right]=\frac{1}{5} \cdot(2-1)^{2}+\frac{3}{5} \cdot(2-2)^{2}+\frac{1}{5} \cdot(2-3)^{2}=\frac{2}{5}$
(c) Let $X_{i}$ be an indicator for the event that the $i$-th ball is blue. What is $\mathrm{E}\left[X_{i}\right]$ ? Find $\mathrm{E}[X]$ again using linearity of expectation.

Solution: The probability that any given ball is blue is 3 out of 6 , namely $1 / 2$. Therefore $\mathrm{E}\left[X_{i}\right]=\mathrm{P}\left(X_{i}=1\right)=1 / 2$ for $i=1,2,3$, and 4 . We get that $\mathrm{E}[X]=\mathrm{E}\left[X_{1}\right]+\mathrm{E}\left[X_{2}\right]+$ $\mathrm{E}\left[X_{3}\right]+\mathrm{E}\left[X_{4}\right]=4 \cdot 1 / 2=2$ confirming the answer in part (a).
2. Roll a 4 -sided die twice. Let $X$ be the larger value and $Y$ be the smaller value. Find (a) the joint PMF of $X$ and $Y$, (b) the marginal PMFs of $X$ and $Y$, (c) $\mathrm{E}[X+Y]$.

## Solution:

(a) When $x>y$ are different, the event $X=x$ and $Y=y$ can happen in 2 out of 36 possible ways: Either the first toss is $x$ and the second toss is $y$, or the other way around. When $x=y$ then there is only one possible outcome. All other probabilities are zero. Summarizing, the joint PMF $p_{X Y}(x, y)$ is

| $x \backslash y$ | 1 | 2 | 3 | 4 |
| :---: | :---: | :---: | :---: | :---: |
| 1 | $1 / 16$ | 0 | 0 | 0 |
| 2 | $1 / 8$ | $1 / 16$ | 0 | 0 |
| 3 | $1 / 8$ | $1 / 8$ | $1 / 16$ | 0 |
| 4 | $1 / 8$ | $1 / 8$ | $1 / 8$ | $1 / 16$ |

(b) The marginal PMFs are obtained by adding the colums and rows, respectively:

| $z$ | 1 | 2 | 3 | 4 |
| :---: | :---: | :---: | :---: | :---: |
| $p_{X}(z)$ | $1 / 16$ | $3 / 16$ | $5 / 16$ | $7 / 16$ |
| $p_{Y}(z)$ | $7 / 16$ | $5 / 16$ | $3 / 16$ | $1 / 16$ |

(c) The expected values are $\mathrm{E}[X]=50 / 16$ and $\mathrm{E}[Y]=30 / 16$. The expected sum is $\mathrm{E}[X+$ $Y]=\mathrm{E}[X]+\mathrm{E}[Y]=5$. Another way to see this is that if $A$ and $B$ are the first and second rolls then $X+Y=A+B$, so $\mathrm{E}[X+Y]=\mathrm{E}[A+B]=\mathrm{E}[A]+\mathrm{E}[B]=2 \cdot 5 / 2=5$.
3. A six-sided die is rolled 6 times.
(a) What is the probability that the value 1 occurs at least once?

Solution: Let $E_{1}$ be this event. Then $E_{1}^{c}$ is the event that no 1 occurs. By independence $\mathrm{P}\left(E_{1}^{c}\right)=(5 / 6)^{6}$ and $\mathrm{P}\left(E_{1}\right)=1-\mathrm{P}\left(E_{1}^{c}\right)=1-(5 / 6)^{6} \approx 0.665$.
(b) Let $N$ be the number of distinct values that occur. For example, if the outcome is 521154, then $N=4$. What is $\mathrm{E}[N]$ ? (Hint: Write $N$ as a sum of indicators)

Solution: Let $E_{i}$ be the event "value $i$ occurred at least once" and $N_{i}$ be the indicator random variable for event $E_{i}$. In part (a) we calculated that $\mathrm{E}\left[N_{1}\right]=\mathrm{P}\left(N_{1}=1\right)=$ $\mathrm{P}\left(E_{1}\right)=1-(5 / 6)^{6}$. The random variables $N_{1}, N_{2}, \ldots, N_{6}$ have the same PMF so $\mathrm{E}\left[N_{i}\right]=1-(5 / 6)^{6}$. By linearty of expectation

$$
\mathrm{E}[N]=\mathrm{E}\left[N_{1}\right]+\mathrm{E}\left[N_{2}\right]+\cdots+\mathrm{E}\left[N_{6}\right]=6 \cdot\left(1-(5 / 6)^{6}\right) \approx 3.991 .
$$

4. Alice can't find her expensive sweater. She estimates that there is a $30 \%$ chance that she left it at the café and a $40 \%$ chance that she left it at the shop (and that it is lost with the remaining probability). The distances between her home, the café, and the shop are given below. On her trip to find the sweater, in which order should she visit the venues so as to minimize her expected round-trip walking distance?


Solution: Let $C$ and $S$ be the events that the sweater is at the café and at the shop, respectively, and $X$ be her walking distance which is a random variable. If Alice walks to the café first, her expected walking distance is

$$
\mathrm{E}[X]=2 \cdot 500 \cdot \mathrm{P}(C)+(500+600+700) \cdot \mathrm{P}\left(C^{c}\right)=1000 \cdot 0.3+1800 \cdot 0.7=1560,
$$

and if she walks to the shop first it is

$$
\mathrm{E}[X]=2 \cdot 700 \cdot \mathrm{P}(S)+(500+600+700) \cdot \mathrm{P}\left(S^{c}\right)=1400 \cdot 0.4+1800 \cdot 0.6=1640
$$

Therefore Alice should try the café first.

