1. Roll a 4 -sided die twice. Let $X$ be the larger number and $Y$ be the smaller number you rolled. Find (a) the conditional PMF of $X$ given $Y$ and (b) $\mathrm{E}[X \mid Y=y]$ for $y=1,2,3,4$.
(a) Solution: In question 2 of homework 1 we derived the joint PMF $p_{X Y}(x, y)$ and the marginal PMF $p_{Y}(y)$. The conditional PMF is $p_{X \mid Y}(x \mid y)=p_{X Y}(x, y) / p_{X}(x)$ :

| $y \backslash x$ | 1 | 2 | 3 | 4 |
| :---: | :---: | :---: | :---: | :---: |
| 1 | $1 / 7$ | $2 / 7$ | $2 / 7$ | $2 / 7$ |
| 2 | 0 | $1 / 5$ | $2 / 5$ | $2 / 5$ |
| 3 | 0 | 0 | $1 / 3$ | $2 / 3$ |
| 4 | 0 | 0 | 0 | 1 |

(b) Solution: We apply the definition of conditional expectation:

$$
\begin{aligned}
& \mathrm{E}[X \mid Y=1]=\frac{1}{7} \cdot 1+\frac{2}{7} \cdot 2+\frac{2}{7} \cdot 3+\frac{2}{7} \cdot 4=\frac{19}{7} \\
& \mathrm{E}[X \mid Y=2]=\frac{1}{5} \cdot 2+\frac{2}{5} \cdot 3+\frac{2}{5} \cdot 4=\frac{16}{5} \\
& \mathrm{E}[X \mid Y=3]=\frac{1}{3} \cdot 3+\frac{2}{3} \cdot 4=\frac{11}{3} \\
& \mathrm{E}[X \mid Y=4]=1 \cdot 4=4
\end{aligned}
$$

2. Cup 1 contains three $\$ 1$ coins. Cup 2 contains a $\$ 1$ coin, a $\$ 2$ coin, and a $\$ 5$ coin. Alice chooses a random cup, takes out two coins, and gives the second coin to Bob.
(a) How many dollars does Bob expect to gain?

Solution: Let $C$ be the chosen cup and $X_{1}, X_{2}$ the values of the first and second coin, respectively. By the total expectation theorem,

$$
\mathrm{E}\left[X_{2}\right]=\frac{1}{2} \mathrm{E}\left[X_{2} \mid C=1\right]+\frac{1}{2} \mathrm{E}\left[X_{2} \mid C=2\right]=\frac{1}{2} \cdot 1+\frac{1}{2} \cdot \frac{1+2+5}{3}=\frac{11}{6} \approx 1.833
$$

(b) Bob sees that the first coin out is a $\$ 1$ coin. Did his expected gain increase?

Solution: Before we calculate let's attempt to reason this out. On the one hand, if the first coin is a $\$ 1$ coin, the more valuable coins remain in play, which suggests that Bob's gain should increase. On the other hand, given that the first coin is a $\$ 1$ coin, Alice is more likely to have chosen cup 1, which suggests that Bob's gain should decrease. It is unclear which of these two effects is more significant.
We apply the total expectation theorem again, this time conditioning on $X_{1}=1$ :

$$
\begin{aligned}
& \mathrm{E}\left[X_{2} \mid X_{1}=1\right]=\mathrm{E}[ \left.X_{2} \mid C=1, X_{1}=1\right] \mathrm{P}\left(C=1 \mid X_{1}=1\right) \\
& \quad+\mathrm{E}\left[X_{2} \mid C=2, X_{1}=1\right] \mathrm{P}\left(C=2 \mid X_{1}=1\right) \\
&=1 \cdot \mathrm{P}\left(C=1 \mid X_{1}=1\right)+\frac{2+5}{2} \cdot \mathrm{P}\left(C=2 \mid X_{1}=1\right)
\end{aligned}
$$

To calculate the conditional probabilities we apply Bayes' rule:

$$
\mathrm{P}\left(C=1 \mid X_{1}=1\right)=\frac{\mathrm{P}\left(X_{1}=1 \mid C=1\right) \cdot 1 / 2}{\mathrm{P}\left(X_{1}=1 \mid C=1\right) \cdot 1 / 2+\mathrm{P}\left(X_{1}=1 \mid C=2\right) \cdot 1 / 2}=\frac{1}{1+1 / 3}=\frac{3}{4}
$$

Therefore

$$
\mathrm{E}\left[X_{2} \mid X_{1}=1\right]=1 \cdot \frac{3}{4}+\frac{7}{2} \cdot \frac{1}{4}=\frac{13}{8}=1.625
$$

The expected gain decreased.
3. Express $X$ and $Y$ below as $1+A+B$ for some Geometric random variables $A$ and $B$. Use this formula to find $\mathrm{E}[X]$ and $\mathrm{E}[Y]$. Are $A$ and $B$ independent? If yes, find $\operatorname{Var}[X]$ and $\operatorname{Var}[Y]$.
(a) The first time $X$ at which both a TH and a HT have appeared in a sequence of coin flips.

Solution: Here $A$ is the time after the first flip it takes for the first pattern to appear, and $B$ is the additional time we have to wait for the second pattern. For example, if the first flip is a H then $A$ is the time of the first TH and $B$ is the additional time it takes for a TH. Then $A$ is the number of flips it takes to get a H and $B$ is the number of flips it takes to get a T . These are Geometric $(1 / 2)$ random variables and they are independent. Therefore $\mathrm{E}[X]=1+\mathrm{E}[A]+\mathrm{E}[B]=1+2+2=5$ and $\operatorname{Var}[X]=\operatorname{Var}[A]+\operatorname{Var}[B]=0+2+2=4$.
(b) The first time $Y$ at which 1, 2, and 3 have all appeared in a sequence of 3 -sided die rolls.

Solution: Here $A$ is the time after the first roll that the second value appears and $B$ is the remaining time it takes for the last value to appear. For example, if the first roll is a 2 , then $B$ is the time we have to wait for a 1 or a 3 . If among these two a 3 appears first then $C$ is the remaining time we have to wait for a 1 . Then $A$ is a Geometric (2/3) random variable and $B$ is a Geometric (1/3) random variable and they are independent. Therefore $\mathrm{E}[Y]=1+\mathrm{E}[A]+\mathrm{E}[B]=1+\frac{3}{2}+3=\frac{11}{2}$ and $\operatorname{Var}[Y]=\operatorname{Var}[A]+\operatorname{Var}[B]=\frac{2 / 3}{(1 / 3)^{2}}+\frac{1 / 3}{(2 / 3)^{2}}=\frac{27}{4}$.
4. You roll a six-sided die and the value is $X_{1}$. You can either cash in $X_{1}$ dollars, or choose to roll again, in which case you cash in the value $X_{2}$ of the second roll in dollars. [Based on Blitzstein-Hwang exercise 4.4]
(a) For which values of $X_{1}$ should you roll again in order to maximize your expected utility?

Solution: The expectation of $X_{2}$ is $\mathrm{E}\left[X_{2}\right]=\frac{1+2+3+4+5+6}{6}=\frac{7}{2}=3.5$ (regardless of the value of $X_{1}$. After observing roll $X_{1}$, you expected utility by cashing in is $X_{1}$, and your expected utility by rolling again is 3.5 . Your best strategy is to cash in if $X_{1}$ is 4,5 , or 6 , and roll again if $X_{1}$ is 1,2 , or 3 .
(b) What is your expected utility for the strategy in part (a)?

Solution: Let $U_{2}$ be the total utility. We condition on the value $x_{1}$ of $X_{1}$. If $x_{1} \geq 4$ the conditional expected utility is $x_{1}$. If $x_{1} \leq 3$ it is $\mathrm{E}\left[X_{2}\right]=7 / 2$. By the total expectation formula

$$
\begin{aligned}
\mathrm{E}\left[U_{2}\right] & =\mathrm{E}\left[U_{2} \mid X_{1} \leq 3\right] \cdot \frac{1}{2}+\mathrm{E}\left[U_{2} \mid X_{1}=4\right] \cdot \frac{1}{6}+\mathrm{E}\left[U_{2} \mid X_{1}=5\right] \cdot \frac{1}{6}+\mathrm{E}\left[U_{2} \mid X_{1}=6\right] \cdot \frac{1}{6} \\
& =\frac{7}{2} \cdot \frac{1}{2}+4 \cdot \frac{1}{6}+5 \cdot \frac{1}{6}+6 \cdot \frac{1}{6} \\
& =4 \frac{1}{4}
\end{aligned}
$$

(c) You can now roll a third time if you are unhappy with $X_{2}$. How does this change your answer in (a) and (b)? (Hint: What is your maximum expected utility given $X_{1}=x_{1}$ ?)

Solution: Given that $X_{1}=x_{1}$, if you decide to roll again, by part (b) your conditional expected utility would be $\mathrm{E}\left[U_{2}\right]=4 \frac{1}{4}$ To maximize your overall expected utility you should therefore take the option if $X_{1} \leq 4$ and stick with your payoff of $X_{1}$ otherwise. By the total expectation theorem the expectation of your utility $U_{3}$ is:

$$
\mathrm{E}\left[U_{3}\right]=\mathrm{E}\left[U_{3} \mid X_{1} \leq 4\right] \cdot \frac{4}{6}+5 \cdot \frac{1}{6}+6 \cdot \frac{1}{6}=4 \frac{1}{4} \cdot \frac{2}{3}+5 \frac{1}{2} \cdot \frac{1}{3}=4 \frac{2}{3}
$$

(d) (Extra credit) What is your maximum expected utility if you can roll up to $t$ times?

Solution: Let $U_{t}$ be the utility. By the same reasoning, your optimal strategy is to take $X_{1}$ if $X_{1}>\mathrm{E}\left[U_{t-1}\right]$ and to roll again if not. When $t=4$ you should roll again unless you get a 5 or a 6 , which gives you expected utility

$$
\mathrm{E}\left[U_{4}\right]=4 \frac{2}{3} \cdot \frac{2}{3}+5 \frac{1}{2} \cdot \frac{1}{3}=4 \frac{17}{18} .
$$

When $t=5$ you should do the same to get

$$
\mathrm{E}\left[U_{5}\right]=4 \frac{17}{18} \cdot \frac{2}{3}+5 \frac{1}{2} \cdot \frac{1}{3}=5 \frac{7}{54} .
$$

When $t=6$ you should roll again unless you get a 6 . Since the expected utility only increases if you are allowed more rolls, this should be your strategy for the first roll for larger $t$ as well. The expected utility is

$$
\mathrm{E}\left[U_{t}\right]=\mathrm{E}\left[U_{t-1}\right] \cdot \frac{5}{6}+6 \cdot \frac{1}{6}=\frac{5}{6} \mathrm{E}\left[U_{t-1}\right]+1 .
$$

We can rewrite this as $\mathrm{E}\left[U_{t}\right]-6=\frac{5}{6}\left(\mathrm{E}\left[U_{t-1}\right]-6\right)$. Iterating it and changing variables $t \rightarrow t+5$ we obtain $\mathrm{E}\left[U_{t+5}\right]-6=\left(\frac{5}{6}\right)^{t}\left(\mathrm{E}\left[U_{5}\right]-6\right)=-\frac{47}{54}\left(\frac{5}{6}\right)^{t}$ from where $\mathrm{E}\left[U_{t+5}\right]=$ $6-\frac{47}{54}\left(\frac{5}{6}\right)^{t}$ for every $t \geq 1$. This gives a formula for the expected utility for all $t$.

