1. N is a Geometric(P) random variable. The success probability P itself is a Uniform random variable independent of N. You observe that $N = 2$. What is the PDF of P given this event? (**Optional**) In general what is the PDF of P given N?

   Solution: By Bayes' rule, conditional PDF of P given $N = 2$ is

   $$f_{P|N}(x|2) = \frac{\mathrm{P}(N = 2|P = p) \cdot f_P(p)}{\mathrm{P}(N = 2)} = \frac{p(1-p) \cdot 1}{\mathrm{P}(N = 2)}.$$

   We calculate $\mathrm{P}(N = 2)$ using the total probability theorem: The denominator of the equation is an integral of the numerator from 0 to 1,

   $$\mathrm{P}(N = 2) = \int_0^1 \mathrm{P}(N = 2|P = p) f_P(p) dp = \int_0^1 (1-p)p dx = \frac{1}{6}.$$

   Hence,

   $$f_{P|N}(p|2) = 6(1-p)p.$$

   By the same reasoning, in general

   $$\begin{aligned} f_{P|N}(p|n) &= \frac{\mathrm{P}(N = n|P = p) f_P(p)}{P(N = n)} \\ &= \frac{(1-p)^{n-1} p}{\int_0^1 (1-p)^{n-1} p dp} \\ &= \frac{(1-p)^{n-1} p}{1/n(n+1)} \\ &= n(n+1)(1-p)^{n-1} p. \end{aligned}$$

2. Let X be an Exponential(λ) random variable. Find the PDF of the random variables (a) $Y = X^2$ and (b) $Z = e^{-\lambda X}$.

   Solution:

   (a) For $y \geq 0$, the CDF of Y is $F_Y(y) = \mathrm{P}(X^2 \leq y) = \mathrm{P}(X \leq \sqrt{y}) = 1 - e^{-\lambda\sqrt{y}}$. The PDF is the derivative of the CDF which is

   $$f_Y(y) = \begin{cases} \frac{\lambda}{2\sqrt{y}} e^{-\lambda\sqrt{y}} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

   (b) Since X only takes nonnegative values, Z will take values between 0 and 1. For $0 < z \leq 1$, the CDF of Z is

   $$F_Z(z) = \mathrm{P}(e^{-\lambda X} \leq z) = \mathrm{P}\Big(X \geq -\frac{\log z}{\lambda}\Big) = e^{-\lambda(-\log z/\lambda)} = z.$$

   Its derivative is

   $$f_Z(z) = \begin{cases} 1 & \text{if } 0 < z \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   In words, Z is a Uniform$(0, 1)$ random variable.

3. Raindrops hit your head at a rate of 1 per second. What is the PDF of the time at which the second raindrop hits you? How about the third one? (**Hint:** convolution)

Solution: The time before the second raindrop is $Y = X_1 + X_2$, where X_1 and X_2 are independent Exponential(1) random variables. We calculate the PDF of Y using the convolution formula:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(x_1) f_{X_2}(y - x_1) dx_1 = \int_0^y e^{-x_1} e^{-y+x_1} dx_1 = ye^{-y}.$$

The third raindrop hits at time $Z = Y + X_3$, where X_3 is another independent Exponential(1) random variable. By the convolution formula again,

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(y) f_{X_3}(z - y) dy = \int_0^z ye^{-y} e^{-z+y} dy = \frac{z^2}{2} e^{-z}.$$

4. The body temperatures of a healthy person and an infected person are Normal$(36.8, 0.5^2)$ and Normal$(37.8, 1.0^2)$ random variables, respectively. About 1% of the population is infected.

(a) What is the conditional probability that I am infected given that my temperature is t?

(b) For which values of t am I more likely to be infected than not?

Solution:

(a) Let A be the event that I am infected, and T be my body temperature. By the total probability theorem,

$$f_T(t) = \mathrm{P}(A) f_{T|A}(x) + \mathrm{P}(A^c) f_{T|A^c}(t),$$

where $T|A$ is a Normal$(37.8, 1.0^2)$ random variable and $T|A^c$ is a Normal$(36.8, 0.5^2)$ random variable. The (unconditional) PDF of X is

$$f_T(t) = \frac{0.01}{\sqrt{2\pi}} e^{-\frac{(t-37.8)^2}{2}} + \frac{0.99}{\sqrt{2\pi}(0.5)} e^{-\frac{(t-36.8)^2}{2(0.5)^2}} = \frac{0.01}{\sqrt{2\pi}} e^{-(t-37.8)^2/2} + \frac{1.98}{\sqrt{2\pi}} e^{-2(t-36.8)^2}.$$

By Bayes' rule, the conditional probability of A given T is

$$P(A|T = t) = \frac{P(A) f_{T|A}(t)}{f_T(t)} = \frac{0.01e^{-(t-37.8)^2/2}}{0.01e^{-(t-37.8)^2/2} + 1.98e^{-2(t-36.8)^2}}$$

(b) I am more likely to be infected than not when $\mathrm{P}(A) > \mathrm{P}(A^c)$, namely when

$$0.01e^{-(t-37.8)^2/2} > 1.98e^{-2(t-36.8)^2}.$$

Taking logarithms of both sides this is equivalent to a quadratic inequality in t. Solving this inequality, we obtain that $\mathrm{P}(A) > \mathrm{P}(A^c)$ holds when $t < t_-$ or $t > t_+$, where $t_- \approx 34.4742$ and $t_+ \approx 38.4591$.