1. A Chicken lays a Poisson $(\lambda)$ number $N$ of eggs. Each egg independently hatches a chick with probability $p$. Let $X$ be the number of chicks that hatch. Calculate
(a) the conditional expectation $\mathrm{E}[X \mid N=n]$;
(b) the unconditional expectation $\mathrm{E}[X]$;
(c) the unconditional expectation $\mathrm{E}[N X]$;
(d) the covariance $\operatorname{Cov}[X, N]$.
[Based on Blitzstein-Hwang Exercise 7.48]

## Solution:

(a) $X$ is a $\operatorname{Binomial}(N, p)$ random variable, so $\mathrm{E}[X \mid N=n]=n p$.
(b) By the total expectation theorem,

$$
\mathrm{E}[X]=\sum_{n=0}^{\infty} \mathrm{E}[X \mid N=n] \mathrm{P}(N=n)=n p \cdot \mathrm{P}(N=n)=\mathrm{E}[N] p=\lambda p
$$

(c) By the total expectation theorem again,

$$
\begin{aligned}
\mathrm{E}[N X] & =\sum_{n=0}^{\infty} \mathrm{E}[N X \mid N=n] \mathrm{P}(N=n) \\
& =\sum_{n=0}^{\infty} n \cdot \mathrm{E}[X \mid N=n] \cdot \mathrm{P}(N=n) \\
& =\sum_{n=0}^{\infty} n \cdot n p \cdot \mathrm{P}(N=n) \\
& =\sum_{n=0}^{\infty} n^{2} \mathrm{P}(N=n) \cdot p \\
& =\mathrm{E}\left[N^{2}\right] p \\
& =\left(\operatorname{Var}[N]+\mathrm{E}[N]^{2}\right) p \\
& =\left(\lambda+\lambda^{2}\right) p
\end{aligned}
$$

(d) $\operatorname{Cov}[X, N]=\mathrm{E}[N X]-\mathrm{E}[N] \mathrm{E}[X]=\left(\lambda+\lambda^{2}\right) p-\lambda \cdot \lambda p=\lambda p$.
2. Six boys and six girls sit randomly at a round table. Let $X$ be the number of boys that sit between two girls.
(a) Let $X_{i}$ be the indicator for the event "boy $i$ sits between two girls." What is $\operatorname{Var}\left[X_{i}\right]$ ?
(b) What is $\operatorname{Cov}\left[X_{i}, X_{j}\right](i \neq j)$ ?
(c) What is $\operatorname{Var}[X]$ ?

## Solution:

(a) The probability that a given boy has a girl to his left is $6 / 11$. Conditioned on this, the probability that he has a girl to his right is $5 / 10=1 / 2$. Therefore the probability that a given boy sits between two girls $\mathrm{P}\left(X_{i}=1\right)=3 / 11$. By the formula for the variance of an indicator random variable, $\operatorname{Var}\left[X_{i}\right]=3 / 11 \cdot 8 / 11=24 / 121$.
(b) By the formula for covariance,

$$
\begin{aligned}
\operatorname{Cov}\left[X_{i}, X_{j}\right] & =\mathrm{E}\left[X_{i} X_{j}\right]-\mathrm{E}\left[X_{i}\right] \mathrm{E}\left[X_{j}\right] \\
& =\mathrm{P}\left(X_{i} X_{j}=1\right)-\mathrm{P}\left(X_{i}=1\right) \mathrm{P}\left(X_{j}=1\right) \\
& =\mathrm{P}\left(X_{j}=1 \mid X_{i}=1\right) \mathrm{P}\left(X_{i}=1\right)-\mathrm{P}\left(X_{i}=1\right) \mathrm{P}\left(X_{j}=1\right)
\end{aligned}
$$

To calculate $\mathrm{P}\left(X_{j}=1 \mid X_{i}=1\right)$, let's call boy $i$ Ivan, boy $j$ John, and the girls that Boy $i$ sits between Alice and Eve. Let $E$ be the event "John sits next to Alice or Eve". Then $\mathrm{P}\left(E \mid X_{i}=1\right)=2 / 9$ because Alice, Ivan, and Eve have already been seated next to one another with Ivan in the middle, and John is equally likely to occupy any of the 9 remaining positions.
Next, $\mathrm{P}\left(X_{j}=1 \mid X_{i}=1, E\right)=4 / 8$ because after the four have been seated next to one another, the vacant seat next to John is equally likely to be occupied by any of the four remaining girls.
Finally, $\mathrm{P}\left(X_{j}=1 \mid X_{i}=1, E^{c}\right)=4 / 8 \cdot 3 / 7$ because after the four have been seated in this formation both seats next to John are still vacant, and the probability they are both occupied by girls is $4 / 8 \cdot 3 / 7$.
We now apply the total probability theorem, conditioning on $E$ :

$$
\begin{aligned}
\mathrm{P}\left(X_{j}=1 \mid X_{i}=1\right)= & \mathrm{P}\left(X_{j}=1 \mid X_{i}=1, E\right) \mathrm{P}\left(E \mid X_{i}=1\right) \\
& \quad+\mathrm{P}\left(X_{j}=1 \mid X_{i}=1, E^{c}\right) \mathrm{P}\left(E^{c} \mid X_{i}=1\right) \\
= & 4 / 8 \cdot 2 / 9+4 / 8 \cdot 3 / 7 \cdot 7 / 9 \\
= & 5 / 18 .
\end{aligned}
$$

Hence, $\operatorname{Cov}\left[X_{i}, X_{j}\right]=5 / 18 \times 3 / 11-(3 / 11)^{2}=1 / 726$.
(c) The variance of $X$ is the sum of the 6 indicator variances $\operatorname{Var}\left[X_{i}\right]$ and the $6 \cdot 5=30$ covariances $\operatorname{Cov}\left[X_{i}, X_{j}\right]$. Therefore $\operatorname{Var}[X]=6 \cdot 24 / 121+30 \cdot 1 / 726=149 / 121 \approx 1.231$.
3. Two typing monkeys sit at special keyboards. The keyboards have only two keys a and b. Each monkey types in a random 200 letter string, independently of the other one. Let $E$ be the event "There is a pattern of 20 consecutive letters that appears in both strings." Is it true that $\mathrm{P}(E)<5 \%$ ? Justify your answer.

Solution: Let's call the two monkeys Alice and Bob. Let $N$ be the number of consecutive 20 -letter strings typed by Alice and Bob that are equal. $N$ is a sum of indicator random variables $N_{a, b}$ for the event "the 20 -letter string starting at position $a$ typed by Alice equals the 20 -letter string starting at position $b$ typed by Bob". Here $a$ and $b$ can take any value between 1 and 181. By linearity of expectation

$$
\mathrm{E}[N]=\mathrm{E}\left[N_{1,1}+N_{1,2}+\cdots+N_{181,181}\right]=\mathrm{P}\left(N_{1,1}=1\right)+\mathrm{P}\left(N_{1,2}=1\right)+\cdots+\mathrm{P}\left(N_{181,181}=1\right) .
$$

No matter which 20 characters $A_{a}, \ldots, A_{a+19}$ were typed by Alice, the probability Bob's 20 characters $B_{b}, \ldots, B_{b+19}$ match them is $2^{-20}$. Therefore all probabilities are equal to $2^{-20}$ and

$$
\mathrm{E}[N]=181^{2} \cdot 2^{-20} \approx 0.031
$$

By Markov's inequality, $\mathrm{P}(N \geq 1) \leq \mathrm{E}[N]$, so the probability of a match is less than $5 \%$.
4. 100 people put their hats in a box and each one pulls a random hat out.
(a) Let $G$ be any 10-person group. What is the probability that everyone in $G$ pulls their own hat?
(b) What is the expected number of 10-person groups in which everyone pulls their own hat?
(c) Show that the probability that 10 or more people pull their own hat is less than $10^{-6}$.

## Solution:

(a) The probability that the first person in the group pulls their own hat is $1 / 100$. Given this happened, the probability that the second person in the group does so is $1 / 99$, and so on. So the probability that everyone in the group succeeds is $1 /(100 \cdot 99 \cdots 91)$.
(b) Let $X_{S}$ be the random variable indicating that everyone in group $S$ pulled their own hat. Then $X$ is the sum of the random variables $X_{S}$. By linearity of expectation, $\mathrm{E}[X]$ is the sum of $\mathrm{E}\left[X_{S}\right]=1 /(100 \cdot 99 \cdots 91)$ over all 10-person groups $S$. As there are $\binom{100}{10}$ ways to choose a 10 -person group,

$$
\mathrm{E}[X]=\binom{100}{10} \cdot \frac{1}{100 \cdot 99 \cdots 91}=\frac{1}{10!}
$$

(c) By Markov's inequality, the probability that at least one group succeeded in pulling all of their own hats is at most

$$
\mathrm{P}(X \geq 1) \leq \frac{\mathrm{E}[X]}{1}=\frac{1}{10!} \approx 2.7557 \times 10^{-7}<10^{-6}
$$

