Practice questions

1. The distance (in metres) of an archer's target from the bull's eye is a random variable with PDF $f(x) = \theta^2 x e^{-\theta x}$ for $x \geq 0$, where the parameter $\theta \geq 0$ measures the archer's skill.

   (a) Bob's first hit is 40 cm away from the bull's eye. What is the maximum likelihood estimate (MLE) of θ?

   (b) Bob's second hit is 20 cm away from the bull's eye. What is the new MLE of θ?

   (c) Alice's two hits are 15cm and 50cm away from the bull's eye. Does the MLE predict that Alice is a more skilled archer than Bob?

2. In Lecture 4 we showed that the maximum likelihood estimator for the mean λ of a Poisson(λ) random variable given that N occurrences are observed within a time unit is N.

   (a) Now subdivide the time unit into 10 equal intervals and suppose that N_i occurrences are observed in the i-th interval. The N_i are then independent samples of a Poisson($\lambda/10$) random variable. What is the maximum likelihood estimator for λ?

   (b) Is the maximum likelihood estimator in part (a) biased or not?

   (c) **(for ESTR)** Can you come up with a sufficient statistic for n samples of a Poisson(λ) random variable?

3. You have a coin that is either always heads ($\theta = 1$) or fair ($\theta = 0$).

   (a) What is the maximum likelihood estimator for θ from n independent coin flips?

   (b) What is *the* unbiased estimator for θ from one coin flip? (There is only one.)

   (c) **(Optional)** Among all unbiased estimators for θ from n coin flips, which one has the smallest variance?

4. You are given three samples of a Zig(θ) random variable, which has PDF

$$f(x) = \begin{cases} 2(x-\theta), & \text{when } \theta \leq x \leq \theta + 1 \\ 0, & \text{otherwise.} \end{cases}$$

   (a) What is the expected value μ of a Zig(θ) random variable?

   (b) Come up with an unbiased estimator for θ that depends only on the sample mean $\overline{X}$.

   (c) Repeat part (b) for the sample maximum MAX.

   (d) (**Optional**) What are the variances of your estimators in (b) and (c)? (**Hint:** Argue that the variance should not depend on θ and assume $\theta = 0$ in the calculation.)

Additional ESTR 2020 questions

5. In this question you will study the bias of estimators for a Geometric(θ) random variable.

   (a) What is the maximum likelihood estimator for θ given a single sample N_1? Is this estimator biased? *[Adapted from textbook problem BT9.1.2]*

   (b) Show that there is exactly one unbiased estimator of θ from N_1. What is it?

   (c) Now suppose that you are given two samples N_1, N_2. Show that $N_1 + N_2$ is a sufficient statistic for θ. (What is the joint PMF of N_1 and N_2 given $N_1 + N_2$?)

   (d) Use part (c) to derive the minimum variance unbiased estimator of θ from N_1 and N_2.

6. You want to estimate the mean μ of a Normal($\mu, 1$) random variable from 100 samples, but 10% of these samples can be corrupted in an arbitrary way. In this setting the sample mean can be wildly inaccurate. Two alternatives are the sample median and the clipped sample mean (sample mean after removing some fraction of outliers). How would you evaluate the relative accuracy of these two estimators (without any assumptions about the locations or values of the corrupted samples)?