Each question is worth 10 points. Explain your answers clearly.

1. You are given one sample X that is either Uniform$(-1, 1)$ if $\Theta = 0$ or Uniform$(0, 4)$ if $\Theta = 1$. Your prior on Θ is equally likely ($\mathrm{P}(\Theta = 0) = \mathrm{P}(\Theta = 1) = 1/2$).

   (a) What is the MAP estimator for Θ from X?

   (b) What is the error probability of the MAP estimator in part (a)?

2. A fair n-sided die with equally likely face values $1, 2, \ldots, n$ is tossed twice.

   (a) What is the maximum likelihood estimator $\hat{N}$ for n given two samples X_1, X_2?

   (b) Let X_3 be the next sample. What is the probability that the next sample X_3 takes one of the values $1, 2, \ldots, \hat{N}$ in the limit as n tends to infinity?

3. A random variable has PMF $f(-1) = f(1) = \theta$, $f(0) = 1 - 2\theta$, where θ is unknown ($0 \le \theta \le \frac{1}{2}$).

   (a) What is the actual standard deviation σ of the random variable?

   (b) What is the PMF of the adjusted sample standard deviation S^2 for two samples?

Each question is worth 10 points. Explain your answers clearly.

1. X is a Normal$(0, \Theta)$ random variable, where the prior PMF of the parameter Θ is $\mathrm{P}(\Theta = 1/2) = 1/2$, $\mathrm{P}(\Theta = 1) = 1/2$. You observe the following three independent samples of X: $1.0, 1.0, -1.0$.
   (a) What is the posterior PMF of Θ?
   (b) What is the MAP estimate of Θ?
2. The true fraction of employees in some company that support longer lunch breaks is 80%. Ten employees are polled about their support for longer lunch breaks (randomly with repetition). Find the probability that at least 70% of the polled employees support longer lunch breaks
   (a) by direct calculation (you may use an online calculator if you provide a reference),
   (b) using an approximation by a normal random variable.
3. A random variable X is Normal$(1, 1)$ with probability p and Normal$(-1, 1)$ with probability $1 - p$, where the parameter p is unknown.
   (a) What is the maximum likelihood estimator of p from a single sample X?
   (b) Is the estimator in part (a) unbiased? Justify your answer.

Each question is worth 10 points. Explain your answers clearly.

1. You are trying to estimate the fraction V of vegetarians in Hong Kong using Bayesian statistics. Your prior is that V is a Uniform$(0, 1/2)$ random variable.

   (a) You poll a random person and they are not a vegetarian. What is the posterior PDF of V?

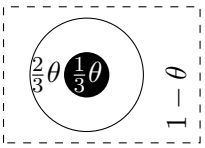
   (b) What is the expected posterior probability that the next polled person will be a vegetarian?

2. A food company produced 1000 boxes of biscuits, 500 of which contain 4 biscuits, 250 contain 3 biscuits and 250 contain one biscuit. You sample two boxes (with repetition) and record the sample mean $\overline{X}$ of the number of biscuits.

   (a) What is the PMF of $\overline{X}$?

   (b) What is the probability that the sample mean equals the actual mean?

3. An archer hits the bull's eye with probability $\frac{1}{3}\theta$, the rest of the target with probability $\frac{2}{3}\theta$, and misses the target with probability $1 - \theta$, where $\theta \in [0, 1]$ is a parameter that models the archer's skill.



   (a) The archer hits the bull's eye twice and misses the board once. What is the maximum likelihood estimate of their skill θ (assuming their shots are independent)?

   (b) Describe an unbiased estimator (3 points) of minimum variance (+2 points) for the player's skill θ from a single attempt. (**Hint:** The estimator assigns a "score" to each outcome.)