A random variable is either uniform on the interval $[0,2]$ (H_0) or has the PDF $f_1(x) = \frac{1}{2}x$ for $0 \le x \le 2$ (H_1). Use the Neyman-Pearson lemma to design a test (for a single sample) with false positive probability 10%, and calculate the false negative probability of this test.

Solution: The PDF for the null hypothesis is $f_0(x) = \frac{1}{2}$ on $[0,2]$ so the likelihood ratio is $f_1(x)/f_0(x) = x$. This is an increasing function of x so by the Neyman-Pearson lemma the optimal test is of the form

$$T(x) = \begin{cases} +, & \text{if } x > t, \\ -, & \text{if } x \le t. \end{cases}$$

The false positive probability is

$$P(T(X) = +|H_0) = \tfrac{1}{2}(2-t).$$

For this to equal 10% we should set $t = 2 - 2\cdot 0.1 = 1.8$. The false negative probability is

$$P(T(X) = -|H_1) = \int_0^t f_1(x)dx = \int_0^{1.8} \tfrac{1}{2}x dx = 0.81.$$