Practice questions

1. Coins A and B have probability of flipping heads equal to 0.01 and 0.1, respectively.

   (a) What is the expected number of heads in 100 tosses for the two coins?

   Solution: These are Binomial$(100, 0.01)$ and Binomial$(100, 0.1)$ random variables A and B, respectively. There expected values are $\mathrm{E}[A] = 100 \cdot 0.01 = 1$ and $\mathrm{E}[B] = 100 \cdot 0.1 = 10$.

   (b) For which one is there a higher chance of observing 5 heads in 100 tosses?

   Solution: The probabilities are

   $$\mathrm{P}[A = 5] = \binom{100}{5} 0.01^5 0.99^{95} \approx 0.003 \quad \text{and} \quad \mathrm{P}[B = 5] = \binom{100}{5} 0.1^5 0.9^{95} \approx 0.034.$$

   Even though A is closer to 5 than B in expectation, the probability of observing the outcome 5 is more than ten times smaller for coin A.

2. Let X and Y be the sum of 30 independent tosses of fair 3-sided dice with face values $\{0, 1, 2\}$ and $\{1, 2, 3\}$, respectively.

   (a) Calculate the means and variances of X and Y.

   Solution: Let X_1 and Y_1 be the value of a single toss of the first and second die, respectively. The mean of X_1 is $\mathrm{E}[X_1] = (0+1+2)/3 = 1$ and $\mathrm{Var}[X_1] = \mathrm{E}[(X_1 - 1)^2] = (1^2 + 0^2 + 1^2)/3 = 2/3$. Since the p.m.f. of Y_1 is the same as the p.m.f. of $X_1 + 1$, we get $\mathrm{E}[Y_1] = \mathrm{E}[X_1] + 1 = 2$ and $\mathrm{Var}[Y_1] = \mathrm{Var}[X_1] = 2/3$. By linearity of expectation, $\mathrm{E}[X] = 30\,\mathrm{E}[X_1] = 30$ and $\mathrm{E}[Y] = 30\,\mathrm{E}[Y_1] = 60$. As the tosses are independent we also get $\mathrm{Var}[X] = 30\,\mathrm{Var}[X_1] = 20$ and $\mathrm{Var}[Y] = 30\,\mathrm{Var}[Y_1] = 20$.

   (b) Apply the Central Limit Theorem to estimate the values $\mathrm{P}(X \geq 45)$ and $\mathrm{P}(Y \geq 45)$.

   Solution: Assuming 30 is sufficiently large, the c.d.f. of X can be approximated by the c.d.f. of a Normal$(30, \sqrt{20})$ random variable, so

   $$\begin{aligned} \mathrm{P}(X \geq 45) &\approx \mathrm{P}(\text{Normal}(30, \sqrt{20}) \geq 45) \\ &= \mathrm{P}(\text{Normal}(0,1) \geq (45 - 30)/\sqrt{20}) \\ &\approx \mathrm{P}(\text{Normal}(0,1) \geq 3.354) \\ &\approx 0.0003 \end{aligned}$$

   Similarly, Y can be approximated by a Normal$(60, \sqrt{20})$ random variable and $\mathrm{P}(Y \geq 45) \approx \mathrm{P}(\text{Normal}(0,1) \geq -3.354) \approx 0.9996$.

   (c) Use normal models for X and Y to estimate the ratio $\mathrm{P}(X = 45)/\mathrm{P}(Y = 45)$.

   Solution: Let f and g be the p.d.f.s of a Normal$(30, \sqrt{20})$ and a Normal$(60, \sqrt{20})$ random variables, respectively. The estimate for the ratio is

   $$\frac{f(45)}{g(45)} = \frac{(2\pi \cdot 20)^{-1/2} e^{-(30-45)^2/2\cdot 20}}{(2\pi \cdot 20)^{-1/2} e^{-(60-45)^2/2\cdot 20}} = 1$$

   As the p.m.f. of Y is the same as the p.m.f. of $X+30$ and both the p.m.f.s of X and Y are symmetric around their mean, $\mathrm{P}(X = 45) = \mathrm{P}(X = 15) = \mathrm{P}(X+30 = 45) = \mathrm{P}(Y = 45)$ so the actual value of the ratio is 1. The normal model estimate is perfectly accurate in this case.

3. A mystery coin is either fair (type 1) or 70% heads, 30% tails (type 2). Your prior belief is that the coin is of type 1 with probability 0.7 and type 2 with probability 0.3. Calculate the posterior probabilities after observing

   (a) a single head,

   Solution: Let Θ denote the coin type. The prior probabilities are $\mathrm{P}(\Theta = 1) = 0.7$ and $\mathrm{P}(\Theta = 2) = 0.3$. Letting H be the event of observing a head, by Bayes' rule $\mathrm{P}(\Theta = 1|H) \propto \mathrm{P}(H|\Theta = 1)\,\mathrm{P}(\Theta = 1) = 0.5 \cdot 0.7$ and $\mathrm{P}(\Theta = 2|H) = \mathrm{P}(H|\Theta = 2)\,\mathrm{P}(\Theta = 2) = 0.7 \cdot 0.3$. The posterior probabilities are

   $$\mathrm{P}(\Theta = 1|H) = \frac{0.5 \cdot 0.7}{0.5 \cdot 0.7 + 0.7 \cdot 0.3} = 0.625$$

   and $\mathrm{P}(\Theta = 2|H) = 1 - \mathrm{P}(\Theta = 1|H) = 0.375$.

   (b) 7 heads in 10 flips.

   Solution: Let N be the number of heads in 10 flips. Then $\mathrm{P}(N = 7|\Theta = 1) = \mathrm{P}(\mathrm{Binomial}(10, 0.5) = 7) \approx 0.117$ so

   $$\mathrm{P}(\Theta = 1|N = 7) \propto \mathrm{P}(N = 7|\Theta = 1)\,\mathrm{P}(\Theta = 1) \approx 0.117 \cdot 0.7 \approx 0.082.$$

   Similarly $\mathrm{P}(N = 7|\Theta = 2) = \mathrm{P}(\mathrm{Binomial}(10, 0.7) = 7) \approx 0.267$ and $\mathrm{P}(\Theta = 2|N = 7) \propto 0.267 \cdot 0.3 \approx 0.080$. The posteriors are

   $$\mathrm{P}(\Theta = 1|N = 7) \approx \frac{0.082}{0.082 + 0.080} \approx 0.506 \quad \text{and}$$
   $$\mathrm{P}(\Theta = 2|N = 7) = 1 - \mathrm{P}(\Theta = 1|N = 7) \approx 0.494.$$

4. Another mystery coin is fair (type 1), all heads (type 2), or all tails (type 3). Assume an equally likely prior on the three types. For any possible sequence of n outcomes derive a formula for the posterior probabilities.

   Solution: Let Θ be the coin type and S be the sequence of outcomes. If S contains both heads and tails then $\mathrm{P}(\Theta = 2|S) = \mathrm{P}(\Theta = 3|S) = 0$ so $\mathrm{P}(\Theta = 1|S) = 1$. If S consists of n heads, then $\mathrm{P}(\Theta = 3|S) = 0$ and

   $$\mathrm{P}(\Theta = 2|S) \propto \mathrm{P}(S|\Theta = 2)\,\mathrm{P}(\Theta = 2) = 1 \cdot 1/3 = 1/3$$

   while

   $$\mathrm{P}(\Theta = 1|S) \propto \mathrm{P}(S|\Theta = 1)\,\mathrm{P}(\Theta = 1) = 2^{-n} \cdot 1/3 = 2^{-n}/3$$

   As the probabilities must add up to one we get that $\mathrm{P}(\Theta = 2|S) = 1/(1 + 2^{-n}) = 2^n/(2^n + 1)$ and $\mathrm{P}(\Theta = 1|S) = 2^{-n}/(1 + 2^{-n}) = 1/(2^n + 1)$. If S consists of n tails, by symmetry we get

   $$\mathrm{P}(\Theta = 1|S) = \frac{1}{2^n + 1}, \quad \mathrm{P}(\Theta = 2|S) = 0, \quad \mathrm{P}(\Theta = 3|S) = \frac{2^n}{2^n + 1}.$$